Solutions

Exam 4 Review: Sections 4.1-4.2

Section 4.1 Find the general form of the solution to the system

$$x' = y$$
$$y' = 2x + y$$

$$x'' = y' = 2x + y = 2x + x'$$

or equivalently $x'' - x' - 2x = 0$.

$r^2 - r - 2 = 0$
$(r-2)(r+1) = 0$ $\Rightarrow$ $x = c_1 e^{2t} + c_2 e^{-t}$

and thus

$$y = x' = 2c_1 e^{2t} - c_2 e^{-t}.$$

Section 4.1 Find the solution to the system

$$x' = 10y,\ y' = -10x;\ x(0) = 3,\ y(0) = 4$$

$$x'' = 10y' = 10(-10x) = -100x$$

or equivalently

$$x'' + 100x = 0$$

$\left.\begin{array}{l} r^2 + 100 = 0 \\ r = \pm 10i \end{array}\right\} \Rightarrow$ $x = c_1 \cos 10t + c_2 \sin 10t$, and thus
$y = x'/10 = -c_1 \sin 10t + c_2 \cos 10t$.

Since $x(0) = 3 = c_1$ and $y(0) = 4 = c_2$, we have the unique solution

$$x = 3\cos 10t$$
$$y = 4\cos 10t$$

Section 4.1 Transform the following differential equations or systems into an equivalent system of first-order differential equations.

(a) $x'' + 3x' + 7x = t^2$

(b) $x'' + 4x - x^3 = 0$

(c) $t^2x'' + tx' + (t^2 - 1)x = 0$

(d) $x'' + 2x' + 26x = 82\cos 4t$

(e) $x'' = (1 - y)x,\ y'' = (1 - x)y$

(a) Let $x_1 = x$ and $x_2 = x'$ and rewrite the equation as

$$x'' = t^2 - 3x' - 7x$$ (optional; see the remaining problems)

Then the equivalent system is

$$x_2 = x_1'$$
$$x_2' = t^2 - 3x_2 - 7x_1$$

(b) Let $x_1 = x$ and $x_2 = x'$.
Then the equivalent system is

$$x_2 = x_1'$$
$$x_2' + 4x_1 - x_1^3 = 0$$

(c) Let $x_1 = x$ and $x_2 = x'$. Then the equivalent system is

$$t^2 x_2' + t x_2 + (t^2 - 1)x_1 = 0$$
$$x_2 = x_1'$$

(d) Let $x_1 = x$ and $x_2 = x'$. Then the equivalent system is

$$x_2' + 2x_2 + 26x_1 = 82\cos 4t$$
$$x_2 = x_1'$$

(e) Let $x_1 = x$ and $x_2 = x'$, and similarly let $y_1 = y$ and $y_2 = y'$.
Then the equivalent system is

$$x_2' = (1 - y_1)x_1$$
$$y_2' = (1 - x_1)y_1$$
$$x_2 = x_1'$$
$$y_2 = y_1'$$

For extra practice you can also attempt to solve these systems.

Section 4.2 Solve the following system of equations

$$x' = x - 2y, \quad y' = 2x - 3y$$

Eqn 1 , Eqn 2.

From Eqn 2, $x = \frac{1}{2}(y' + 3y)$ and hence $x' = \frac{1}{2}(y'' + 3y')$. From Eqn 1, we also have $x' = x - 2y = \frac{1}{2}(y' + 3y) - 2y = \frac{1}{2}(y' - y)$.

Setting them equal, we get

$$\frac{1}{2}(y'' + 3y') = \frac{1}{2}(y' - y)$$

$$y'' + 3y' = y' - y$$

$$y'' + 2y' + y = 0$$

$$\left.\begin{array}{l} r^2 + 2r + 1 = 0 \\ (r+1)^2 = 0 \end{array}\right\} \Rightarrow y = (c_1 + c_2 t)e^{-t}.$$

and thus $y' = (c_2 - c_1)e^{-t} - c_2 t e^{-t}$

Combining with Eqn 2, we solve for x:

$$x = \frac{1}{2}(y' + 3y)$$

$$= \frac{1}{2}\left[(c_2 - c_1)e^{-t} - c_2 t e^{-t} + 3(c_1 + c_2 t)e^{-t}\right]$$

$$= \frac{1}{2}\left[(c_2 + 2c_1)e^{-t} + 2c_2 t e^{-t}\right].$$

Section 4.2 Solve the following system of equations

$$x' = -3x + 2y, \; y' = -3x + 4y; \; x(0) = 0, \; y(0) = 2$$

Eqn 1 , Eqn 2

From Eqn 1, $y = \frac{1}{2}(x' + 3x)$ and hence $y' = \frac{1}{2}(x'' + 3x')$.

From Eqn 2, we also have $y' = -3x + 4y = -3x + 4 \cdot \frac{1}{2}(x' + 3x) = 3x + 2x'$.

Setting them equal, we get

$$\frac{1}{2}(x'' + 3x') = 3x + 2x'$$

$$x'' + 3x' = 6x + 4x'$$

or equivalently $x'' - x' - 6x = 0$.

$$\left.\begin{array}{l} r^2 - r - 6 = 0 \\ (r-3)(r+2) = 0 \end{array}\right\} \Rightarrow x = c_1 e^{3t} + c_2 e^{-2t} \text{ and hence } x' = 3c_1 e^{3t} - 2c_2 e^{-2t}.$$

Combining with Eqn 1, we can also solve for y:

$$y = \frac{1}{2}(x' + 3x)$$

$$= \frac{1}{2}\left(3c_1 e^{3t} - 2c_2 e^{-2t} + 3(c_1 e^{3t} + c_2 e^{-2t})\right)$$

$$= \frac{1}{2}\left(6c_1 e^{3t} + c_2 e^{-2t}\right)$$

$$= 3c_1 e^{3t} + \frac{1}{2} c_2 e^{-2t}.$$

Next we use the initial values to solve for c_1 and c_2.

$$\left.\begin{array}{l} (1)\; x(0) = 0 = c_1 + c_2 \\ (2)\; y(0) = 2 = 3c_1 + \frac{1}{2}c_2 \end{array}\right\} \Rightarrow \begin{array}{l} (1) - 2(2) \\ -4 = -5c_1 \end{array} \Rightarrow c_1 = \frac{4}{5}$$

and thus $c_2 = -\frac{4}{5}$.

Therefore our unique solution is

$$x(t) = \frac{4}{5}\left(e^{3t} - e^{-2t}\right)$$

$$y(t) = \frac{4}{5}\left(3e^{3t} - \frac{1}{2}e^{-2t}\right).$$

Section 4.2 Solve the following system of equations

$$x' - 4x + 3y = 0$$
$$-6x + y' + 7y = 0$$

For the remaining problems you can use the method from the problems on the previous page, but I will solve them using the differential operator D; i.e. $D[f] = f'$.

First rewrite the system as
$$(D-4)x + 3y = 0 \quad (1)$$
$$-6x + (D+7)y = 0 \quad (2)$$

Then multiply (1) by -6 and (2) by $(D-4)$ to get
$$-6(D-4)x - 18y = 0 \quad (3)$$
$$-6(D-4)x + (D^2+3D-28)y = 0 \quad (4)$$

Subtract (3) from (4) to get
$$0 = [(D^2+3D-28)+18]y = (D^2+3D-10)y = y'' + 3y' - 10y. \quad (7)$$

Similarly you can multiply (1) by $(D+7)$ and (2) by 3 to get
$$(D^2+3D-28)x + 3(D+7)y = 0 \quad (5)$$
$$-18x \quad + 3(D+7)y = 0 \quad (6)$$

and subtract (5) from (6) to get
$$0 = [-18 - (D^2+3D-28)]x = -(D^2+3D-10)x = -x'' - 3x' + 10x \quad (8)$$

Solving (7) and (8) using characteristic equations yields
$$x = a_1 e^{2t} + a_2 e^{-5t}$$
$$y = b_1 e^{2t} + b_2 e^{-5t}$$
and solve for the relations to get
$$x(t) = \frac{3}{2} b_1 e^{2t} + \frac{1}{3} b_2 e^{-5t}$$
$$y(t) = b_1 e^{2t} + b_2 e^{-5t}.$$

You could also just solve Eqn (7) for y and use the method of elimination from the previous two problems to solve for x.

On the next problem I will use the same method as this problem, but in a smarter way using linear algebra.

Section 4.2 Solve the following system of equations

$$x' = 2x + y$$
$$y' = x + 2y - e^{2t}$$

Rearrange so that

$$(D-2)x + (-1)y = 0$$
$$(-1)x + (D-2)y = -e^{2t}$$

Let $\bar{x} = \begin{pmatrix} D-2 \\ -1 \end{pmatrix}$, $\bar{y} = \begin{pmatrix} -1 \\ D-2 \end{pmatrix}$ and $\bar{f} = \begin{pmatrix} 0 \\ -e^{2t} \end{pmatrix}$.

Solve for the determinants of $(\bar{x}, \bar{y})$, $(\bar{x}, \bar{f})$ and $(\bar{f}, \bar{y})$:

$$\det(\bar{x}, \bar{y}) = \det\begin{pmatrix} D-2 & -1 \\ -1 & D-2 \end{pmatrix} = (D-2)^2 - (-1)^2 = D^2 - 4D + 3$$

$$\det(\bar{x}, \bar{f}) = \det\begin{pmatrix} D-2 & 0 \\ -1 & -e^{2t} \end{pmatrix} = (D-2)(-e^{2t}) - (-1)\cdot 0 = (-2e^{2t} + 2e^{2t}) = 0$$

$$\det(\bar{f}, \bar{y}) = \det\begin{pmatrix} 0 & -1 \\ -e^{2t} & D-2 \end{pmatrix} = 0(D-2) - (-1)(-e^{2t}) = e^{2t}$$

Finally, you solve for x and y. First x: $(D^2 - 4D + 3)x = e^{2t}$

$$x'' - 4x' + 3x = e^{2t}$$

So $x = a_1 e^{3t} + a_2 e^{t} + A e^{2t} = x_c + x_p$

$$= a_1 e^{3t} + a_2 e^{t} + e^{2t}.$$

Similarly for y:

$$(D^2 - 4D + 3)y = 0$$

$$y'' - 4y' + 3y = 0$$

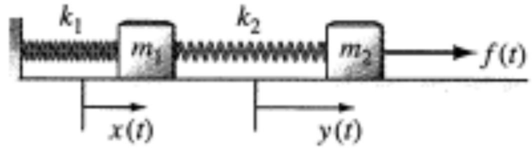
So $y = b_1 e^{3t} + b_2 e^{t}$. Using the original equations we get $b_1 = a_1$ & $b_2 = -a_2$

So that

$$x = a_1 e^{3t} + a_2 e^{t} + e^{2t}$$

$$y = a_1 e^{3t} - a_2 e^{t}.$$

Section 4.2 Consider the following diagram:



Equilibrium positions

Whenever $k_1 = 4,\ k_2 = 2,\ m_1 = 2,\ m_2 = 1$ and $f(t) = 0$ the system becomes

$$x'' + 3x - y = 0$$
$$-2x + y'' + 2y = 0$$

Find the general form of this system of equations.

$$(D^2+3)x + (-1)y = 0$$
$$-2x + (D^2+2)y = 0.$$

Let $\bar{x} = \begin{pmatrix} D^2+3 \\ -2 \end{pmatrix}$, $\bar{y} = \begin{pmatrix} -1 \\ D^2+2 \end{pmatrix}$ and $\bar{f} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Then $\det(\bar{x}, \bar{y}) = \det\begin{pmatrix} D^2+3 & -1 \\ -2 & D^2+2 \end{pmatrix} = (D^2+3)(D^2+2) - (-1)(-2) = D^4 + 5D^2 + 4 = (D^2+1)(D^2+4)$

and $\det(\bar{f}, \bar{y}) = \det\begin{pmatrix} 0 & -1 \\ 0 & D^2+2 \end{pmatrix} = 0 \cdot (D^2+2) - (-1) \cdot 0 = 0.$

So
$$(D^2+1)(D^2+4)x = 0 = x^{(4)} + 5x^{(2)} + 4x.$$

The characteristic eqn has roots $\pm i, \pm 2i$, so

$$x = a_1 \cos t + a_2 \sin t + b_1 \cos 2t + b_2 \sin 2t.$$

From Eqn 1,
$$y = x'' + 3x = (-a_1 \cos t - a_2 \sin t - 4b_1 \cos 2t - 4b_2 \sin 2t)$$
$$+ 3(a_1 \cos t + a_2 \sin t + b_1 \cos 2t + b_2 \sin 2t)$$

$$y = 2(a_1 \cos t + a_2 \sin t) - (b_1 \cos 2t + b_2 \sin 2t)$$